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Electrohydrodynamics of a pair of liquid drops

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In this paper we consider the flow field generated by a uniform electrostatic field in and about a pair of identical liquid drops immersed in a conducting fluid. It is assumed that the undisturbed electric field is parallel to the line joining the centres of the two drops. The flow field is due to the tangential electric stress over the surfaces of the drops and here this stress and the flow field are expressed in terms of bispherical harmonics. When the distance between the centres of the drops is many drop diameters the tangential electric stress and the flow field in and about one drop are unaffected by the presence of the other drop, as expected. When the distance between the centres of the drops is of the order of two drop diameters there is a substantial modification in the tangential electric stress at the surfaces of the drops and in the induced flow field, especially in the region between the planes through the drop centres perpendicular to the undisturbed electric field.

1. Introduction

The mechanics of the interaction of a d.c. electric field with a liquid drop immersed in an insulating liquid are well known. The drop becomes elongated in the direction of the field and usually bursts at high fields (Garton & Krasucki 1964). When the liquid in which the drop is immersed is conducting, at the drop surface there is an imbalance in the tangential component of the electric field stress and this generates a flow field in the drop and its surroundings (Taylor 1966). This was confirmed experimentally by Torza, Cox & Mason (1971), who investigated also the case where the applied field is an a.c. field. The development of the flow field about the drop and its surroundings was considered by Sozou (1973).

The axisymmetric problem of two identical conducting drops immersed in an insulating liquid with the line joining their centres along the direction of the impressed d.c. field was investigated theoretically and experimentally by Latham & Roxburgh (1966) and by Brazier-Smith, Jennings & Latham (1971). The drops again become elongated in the direction of the impressed field but now the bursting occurs at lower fields. The interaction of an electric field with two liquid drops in a conducting fluid has not been investigated either theoretically or experimentally. The subject of this paper is the theoretical investigation of this problem for a d.c. field. For simplicity we restrict our analysis to the axisymmetric case of two identical drops whose surfaces remain approximately spherical. This enables us to express the electrostatic potential and velocity field in terms of series of bispherical harmonics. The closer the drops are, the larger the number of harmonics needed for an accurate evaluation of the electrostatic field **E** and flow field **v**. In our computations we used sufficient harmonics to enable us to express **E** and **v** accurately for cases where the distance between the centres of the drops is 1·1 drop diameters. For experimental testing of the theory set out in the following sections, the drops, when fairly close to each other, must be supported because otherwise, as pointed out by Taylor (1968), the attraction between them will destroy the assumed equilibrium configuration.

2. Electromagnetic equations

We consider two identical liquid drops, assumed spherical, immersed in an incompressible viscous conducting fluid. We use bispherical polar co-ordinates (ξ, η), which for axisymmetric configurations are related to cylindrical polars (r, θ, z) by

$$z = \frac{a \sinh \xi}{\cosh \xi - \cos \eta}, \quad r = \frac{a \sin \eta}{\cosh \xi - \cos \eta}, \tag{1}$$

where a is a constant, the z axis is along the line joining the centres of the drops and the plane $z = 0$ bisects the line joining the centres of the two drops. The plane $z = 0$ corresponds to $\xi = 0$ and the surfaces of the two drops correspond to $\xi = \xi_0$ and $\xi = -\xi_0$. The interiors of the drops correspond to

$$\xi_0 < \xi \leqslant \infty \quad \text{and} \quad -\infty \leqslant \xi < -\xi_0,$$

respectively. The radius R_0 of each drop is given by

$$R_0 = a \operatorname{cosech} \xi_0, \tag{2}$$

and the centres of the drops are at $z = a \coth \xi_0$ and $z = -a \coth \xi_0$, respectively. Owing to the overall symmetry of the problem we need only consider the half-space $z \geqslant 0$, that is $\xi \geqslant 0$.

We assume that the system is subjected to a uniform electric field which at infinity is parallel to the z axis and has magnitude E_0. If, for the half-space $\xi \geqslant 0$, we let the suffix 1 refer to the drop and the suffix 2 to the fluid surrounding it, it can easily be shown that for $\xi > 0$ the electrostatic potential Φ is given by

$$\Phi_1 = E_0 (\cosh \xi - \mu)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n \exp\left[-(n+\tfrac{1}{2})\,\xi\right] P_n(\mu), \tag{3}$$

$$\Phi_2 = E_0 (\cosh \xi - \mu)^{\frac{1}{2}} \sum_{n=0}^{\infty} B_n \sinh\left[(n+\tfrac{1}{2})\xi\right] P_n(\mu) - E_0 z, \tag{4}$$

where $\mu = \cos \eta$, A_n and B_n are constants to be determined and $P_n(\mu)$ is the Legendre polynomial of degree n. For $\xi > 0$, z may be expressed in the form

$$z = 2^{\frac{1}{2}} a (\cosh \xi - \mu)^{\frac{1}{2}} \sum (2n+1) \exp\left[-(n+\tfrac{1}{2})\,\xi\right] P_n(\mu). \tag{5}$$

The electric field **E** is related to Φ by $\mathbf{E} = -\nabla \Phi$. At the surface of the drop we must have continuity of Φ (or continuity of the tangential component of **E**) and continuity of the normal component of the electric current; that is, continuity

$\cosh\xi_0$	λ	$-a_0$	$-a_1$	$-a_2$	$-a_3$	$-a_{24}$
4	10	0·364	0·069	0·012	0·002	$1\cdot5\times10^{-21}$
4	0·05	0·352	0·177	0·039	0·007	$7\cdot9\times10^{-21}$
1·5	10	0·723	0·404	0·199	0·093	$1\cdot1\times10^{-7}$
1·5	0·05	0·560	0·870	0·601	0·331	$3\cdot9\times10^{-7}$
1·1	10	1·207	1·165	0·920	0·695	$5\cdot6\times10^{-4}$
1·1	0·05	0·641	1·595	1·959	1·892	$1\cdot2\times10^{-3}$
1·05	10	1·459	1·662	1·478	1·249	$1\cdot5\times10^{-2}$
1·05	0·05	0·655	1·772	2·479	2·769	$2\cdot6\times10^{-2}$

TABLE 1. Values of a_0, a_1, a_2, a_3 and a_{24} for some ξ_0, for $\lambda = 10$ and $\lambda = 0\cdot05$

of the normal component of $\sigma\mathbf{E}$, where σ denotes electrical conductivity. From (3)–(5) it follows that the continuity of Φ at $\xi = \xi_0$ requires that

$$A_n \exp\left[-(n+\tfrac{1}{2})\xi_0\right] = B_n \sinh(n+\tfrac{1}{2})\xi_0 - 2^{\frac{1}{2}}a(2n+1)\exp\left[-(n+\tfrac{1}{2})\xi_0\right] \qquad (6)$$

and the continuity of the normal component of $\sigma\nabla\Phi$ requires that

$$\lambda\,\partial\Phi_1/\partial\xi = \partial\Phi_2/\partial\xi, \qquad (7)$$

where $\lambda = \sigma_1/\sigma_2$. If we now multiply both sides of (7) by $\cosh\xi_0 - \mu$ and, making use of (6) and the relationship

$$(2n+1)\,\mu P_n(\mu) = (n+1)\,P_{n+1} + nP_{n-1}, \qquad (8)$$

equate coefficients of P_m $(m = 0, 1, 2, 3, \ldots)$ on the two sides of the resulting expression, after a little rearrangement we obtain the following set of equations:

$$\begin{aligned}(n+1)\,a_{n+1}(\lambda+c_{n+1}) &+ a_n[(\lambda-1)\sinh\xi_0 - (2n+1)(\lambda+c_n)\cosh\xi_0] \\ &+ na_{n-1}(\lambda+c_{n-1}) = -2^{\frac{1}{2}}a[(n+1)(2n+3)(1+c_{n+1}) - (2n+1)^2(1+c_n)\,e^{\xi_0} \\ &+ n(2n-1)(1+c_{n-1})\,e^{2\xi_0}]\exp\left[-(n+\tfrac{3}{2})\xi_0\right] \quad (n \geqslant 1),\end{aligned} \qquad (9)$$

where $a_n = A_n \exp[-(n+\frac{1}{2})\xi_0]$, $c_n = \coth(n+\frac{1}{2})\xi_0$ and $a_{-1} = 0$.

The first m equations of the set (9) contain the $m+1$ unknowns $a_0, a_1, a_2, \ldots, a_m$. We have solved this set on the assumption that $a_m = 0$; that is, we have solved the first m equations of this set for $a_0, a_1, \ldots, a_{m-1}$. The quantity $\cosh\xi_0$ is the ratio of the distance between the centres of the drops to a drop diameter. Thus the smaller ξ_0 is the larger the number of a's required and therefore the number of equations to be solved for an accurate evaluation of the electric field. For computational uniformity we set $a_{25} = 0$ and solved the set (9) for all values of ξ_0 considered. Table 1 shows values of a_0, a_1, a_2, a_3 and a_{24} for some ξ_0 for the cases $\lambda = 10$ and $\lambda = 0\cdot05$. For large and moderate values of ξ_0, say $\xi_0 \geqslant 2$, the $|a|$'s decrease monotonically and rapidly and a_{24} is negligible. As ξ_0 decreases all the $|a|$'s increase but $|a_n|$ increases faster than $|a_{n-1}|$. Thus, as ξ_0 decreases, at some stage $|a_n|$ begins to increase with n until it reaches a maximum and thence as n increases $|a_n|$ decreases. For example when $\cosh\xi_0 = 1\cdot5$ and $\lambda = 0\cdot05$, $|a_1|$ is the largest of the $|a|$'s and when $\cosh\xi_0 = 1\cdot05$ and $\lambda = 0\cdot05$, $|a_3|$ is the largest.

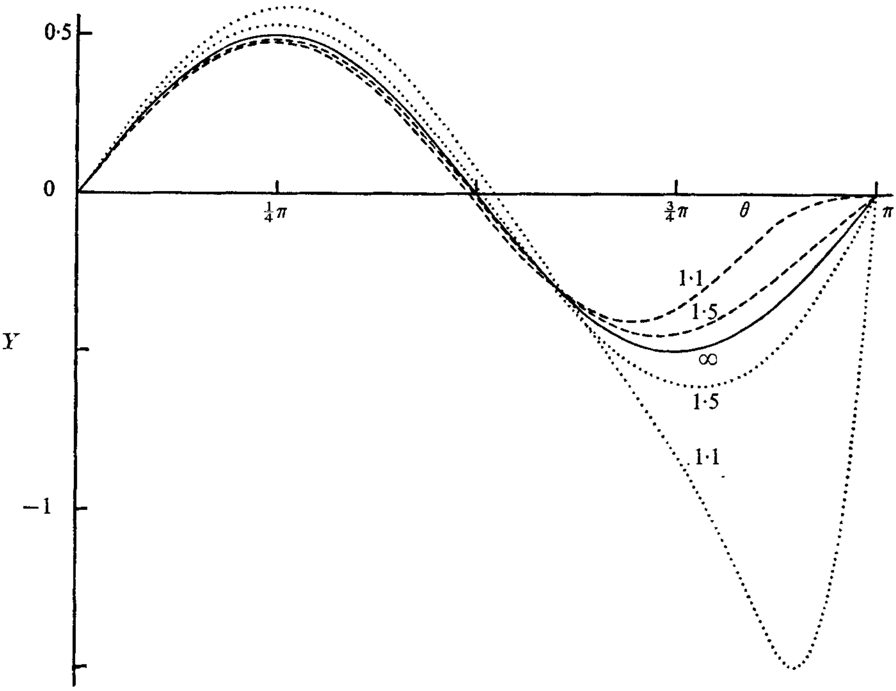

FIGURE 1. Values of $(p_{\xi\eta})_E$ on the sphere $\xi = \xi_0$ as functions of θ, the angle between the positive-z axis and the radius from the centre to the surface of the drop,

$$Y = (2+\lambda)^2 (p_{\xi\eta})_E / 9\epsilon_0 E_0(\kappa_1 - \lambda\kappa_2).$$

$\cdots\cdots$, $\lambda = 10$; ----, $\lambda = 0{\cdot}05$; ——, one drop only. The numbers on the curves are values of ξ_0.

Since $\mathbf{E} = -\nabla\Phi$, it is obvious from (3) and table 1 that, if we use only the first twenty-five A's occurring in (3) in evaluating $\mathbf{E}$, our results will be reasonably accurate when $\cosh\xi_0 \geqslant 1{\cdot}1$. For evaluating $\mathbf{E}$ accurately for $\cosh\xi_0 < 1{\cdot}1$, we may have to make use of more than the first twenty-five A's occurring in (3). This was confirmed by detailed calculations. When we increased the number of A's used to evaluate $\mathbf{E}$ to 29, $\mathbf{E}\,(\xi_0, \mu)$ was practically unaffected for $\xi_0 \geqslant \cosh^{-1} 1{\cdot}1$, whereas when $\xi_0 = \cosh^{-1} 1{\cdot}05$, $\mathbf{E}(\xi_0, \mu)$ had not yet converged to a limit.

The tangential stress $(p_{\xi\eta})_E$ exerted on the surface of the drop $\xi = \xi_0$ by the electric field $\mathbf{E}$ is given by

$$(p_{\xi\eta})_E = \epsilon_0 E_\eta[\kappa_1 E_{1\xi} - \kappa_2 E_{2\xi}], \tag{10}$$

where ϵ_0 is the permittivity of free space, κ the dielectric constant and E_η and E_ξ the tangential and normal components, respectively, of the electric field on the surface $\xi = \xi_0$. On making use of (7), (10) becomes

$$(p_{\xi\eta})_E = -\frac{\epsilon_0}{a^2}(\kappa_1 - \lambda\kappa_2)(1-\mu^2)^{\frac{1}{2}}(\cosh\xi - \mu)^2 \frac{\partial\Phi_1}{\partial\mu}\frac{\partial\Phi_1}{\partial\xi}. \tag{11}$$

When only one drop is present, corresponding to $a\,\text{cosech}\,\xi_0 = R_0$ as $\xi_0 \to \infty$,

$$(p_{\xi\eta})_E = 9\epsilon_0 E_0^2 (2+\lambda)^{-2} (\kappa_1 - \lambda\kappa_2)\sin\theta\cos\theta, \tag{11a}$$

where θ is the angle between the undisturbed electric field and the radius from the centre of the sphere to the drop surface. Figure 1 shows values of $(p_{\xi\eta})_E$ for some ξ_0 for the cases $\lambda = 10$ and $\lambda = 0{\cdot}05$.

3. The flow field

As pointed out by Taylor (1966), the tangential electric stress at the drop surfaces generates a flow field such that the tangential hydrodynamic stress associated with it balances $(p_{\xi\eta})_E$ at $\xi = \pm\xi_0$. The imbalance in $p_{\xi\xi}$, the component of the stress (hydrodynamic and electric) normal to the drop surface, is compensated by the surface tension associated with a suitable drop deformation. Here we assume that the surface tension is sufficiently large so that the drop deformations are small and the drop surfaces are approximately spherical. We are not going to estimate drop deformations here and thus we do not need to evaluate $p_{\xi\xi}$.

Following Taylor (1966) we assume that the velocity field is small and ignore the convection of the electrostatic surface charge by the hydrodynamics currents; that is, we assume that the electrostatic potential calculated in §2 is unaffected by the induced motion. We also assume that the inertia terms in the momentum equation are negligible in comparison with the viscous ones. Thus our steady-state momentum equation becomes

$$\nabla p = \nu\rho\nabla^2\mathbf{v}, \tag{12}$$

where p denotes pressure, ν a coefficient of kinematic viscosity and ρ density.

Owing to the overall geometry of the problem the velocity field is obviously axisymmetric and in terms of a stream function ψ

$$\mathbf{v} = (v_\xi, v_\eta) = -\frac{(\cosh\xi-\mu)^2}{a^2}\left[\frac{\partial\psi}{\partial\mu}, \frac{1}{(1-\mu^2)^{\frac{1}{2}}}\frac{\partial\psi}{\partial\xi}\right]. \tag{13}$$

A suitable form of ψ such that the corresponding velocity field satisfies (12) is (Stimson & Jeffery 1926)

$$\psi = (\cosh\xi-\mu)^{-\frac{3}{2}}(1-\mu^2)\sum_{n=1}^{\infty} U_n(\xi)\,P_n'(\mu), \tag{14}$$

where
$$U_n(\xi) = a_n'\exp\left[-(n-\tfrac{1}{2})\,\xi\right]+b_n'\exp\left[(n-\tfrac{1}{2})\,\xi)\right]+c_n'\exp\left[-(n+\tfrac{3}{2})\,\xi\right] + d_n'\exp\left[(n+\tfrac{3}{2})\,\xi\right]. \tag{15}$$

The constants a_n', b_n', c_n' and d_n' are determined from the boundary conditions. Since the velocity field is finite within the drop, where ξ may tend to infinity, we must have

$$b_{1n}' = d_{1n}' = 0. \tag{16}$$

The symmetry about the plane $\xi = 0$ of $(p_{\xi\eta})_E$, which generates the flow field, requires that

$$a_{2n}'+b_{2n}' = 0, \quad c_{2n}'+d_{2n}' = 0. \tag{17}$$

At the drop surface the normal component of the velocity is zero and the tangential component is continuous, that is

$$U_{1n}(\xi_0) = U_{2n}(\xi_0) = 0, \quad U_{1n}'(\xi_0) = U_{2n}'(\xi_0). \tag{18), (19}$$

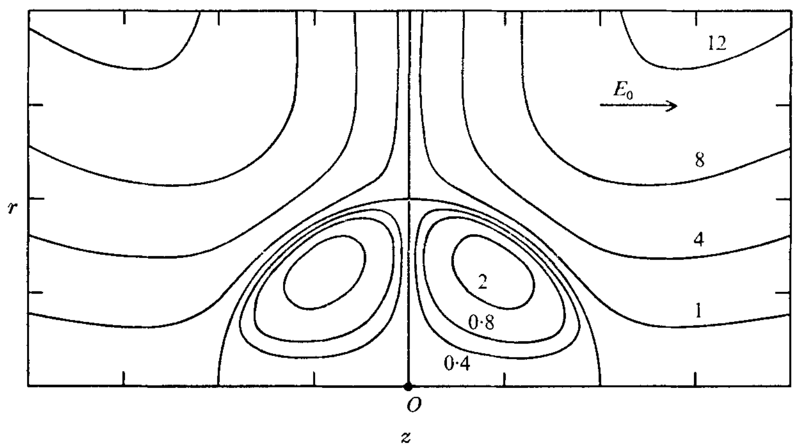

FIGURE 2. Streamlines in the upper half-plane of a meridian section for the case of only one drop. The numbers on the curves are values of $40|\psi|/A$.

From (15)–(19) we deduce that

$$U_{1n}=C_n\{\exp[-(n-\tfrac{1}{2})(\xi-\xi_0)]-\exp[-(n+\tfrac{3}{2})(\xi-\xi_0)]\}, \tag{20}$$

$$U_{2n} = 4C_n[\sinh(n-\tfrac{1}{2})\xi\sinh(n+\tfrac{3}{2})\xi_0-\sinh(n+\tfrac{3}{2})\xi\sinh(n-\tfrac{1}{2})\xi_0]/D_n, \tag{21}$$

where
$$D_n = (2n-1)\sinh 2\xi_0-4\sinh(n-\tfrac{1}{2})\xi_0\cosh(n+\tfrac{3}{2})\xi_0, \tag{22}$$

and the constants C_n are evaluated from balancing the tangential stress $p_{\xi\eta}$ (both hydrodynamic and electric components) across the surface $\xi=\xi_0$. The hydrodynamic part of $p_{\xi\eta}$, say $(p_{\xi\eta})_H$, is given by

$$\frac{a}{\nu\rho}(p_{\xi\eta})_H = \frac{\partial}{\partial\xi}[(\cosh\xi-\mu)v_\eta]-(1-\mu^2)^{\frac{1}{2}}\frac{\partial}{\partial\mu}[(\cosh\xi-\mu)v_\xi]. \tag{23}$$

On making use of (13), (14) and (18) we find that at $\xi=\xi_0$

$$\frac{a^3}{\nu\rho}(p_{\xi\eta})_H = -(\cosh\xi_0-\mu)^{\frac{3}{2}}(1-\mu^2)^{\frac{1}{2}}\sum_1^\infty U_n''(\xi_0)P_n'(\mu). \tag{24}$$

Since at $\xi=\xi_0$
$$(p_{\xi\eta})_E+(p_{\xi\eta})_{2H}-(p_{\xi\eta})_{1H}=0, \tag{25}$$

when we make use of (11) and (24) we obtain

$$\sum_{n=1}^\infty[\nu_2\rho_2U_{2n}''(\xi_0)-\nu_1\rho_1U_{1n}''(\xi_0)]P_n'(\mu) = -a\epsilon_0(\kappa_1-\lambda\kappa_2)(\cosh\xi-\mu)^{\frac{1}{2}}\frac{\partial\Phi_1}{\partial\xi}\frac{\partial\Phi_1}{\partial\mu}.$$

Hence

$$\begin{aligned}\nu_2\rho_2U_{2n}''(\xi_0)-\nu_1\rho_1U_{1n}''(\xi_0) &= -f_n(\xi_0)\\ &= -a\epsilon_0(\kappa_1-\lambda\kappa_2)\frac{(2n+1)}{2n(n+1)}\int_{-1}^1(\cosh\xi_0-\mu)^{\frac{1}{2}}(1-\mu^2)\frac{\partial\Phi_1}{\partial\xi}\frac{\partial\Phi_1}{\partial\mu}d\mu.\end{aligned} \tag{26}$$

From (20)–(22) and (26) we can determine C_n, which turns out to be given by

$$C_n = f_nD_n/2(2n+1)\nu_1\rho_1\{4[M\sinh(n+\tfrac{3}{2})\xi_0+\cosh(n+\tfrac{3}{2})\xi_0]\sinh(n-\tfrac{1}{2})\xi_0 - (2n-1)\sinh 2\xi_0\},$$

where $M=\nu_2\rho_2/\nu_1\rho_1$. Hence we can determine $U_n(\xi)$ and the flow field.

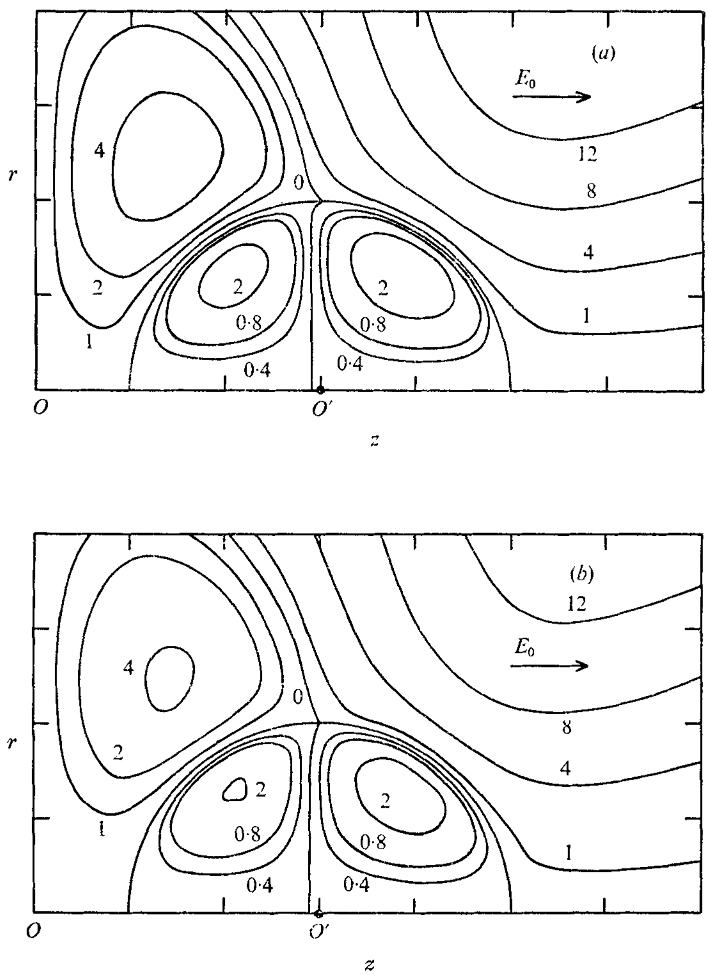

FIGURE 3. Streamlines in and about the drop $\xi_0 = \cosh^{-1} 1{\cdot}5$, in the first quadrant of a meridian section of the r, z plane, for the case $M = 5$. The numbers on the curves are values of $40|\psi|/A$. The centre of the drop is at O'. (*a*) $\lambda = 10$. (*b*) $\lambda = 0{\cdot}05$.

When only one drop is present $(p_{\xi\eta})_E$, given by (11*a*), is symmetric about the plane through the centre of the drop perpendicular to the undisturbed electric field $\mathbf{E}$ and therefore so must be the flow field generated. For this particular case ψ, in terms of spherical polar co-ordinates (R, θ, ϕ) with the origin at the centre of the drop and the axis $\theta = 0$ along the undisturbed electric field, is given by (Taylor 1966)

$$\psi = \begin{cases} A[(R/R_0)^2 - (R/R_0)^5] \sin^2\theta\cos\theta, & R_0 > R, \\ A[(R_0/R)^2 - 1] \sin^2\theta\cos\theta, & R_0 < R, \end{cases} \tag{27}$$

where $A = 0{\cdot}9\, \epsilon_0 E_0^2(\kappa_1 - \lambda\kappa_2)/\nu_1\rho_1(1+M)\,(2+\lambda)^2$. The flow field for this particular case is shown in figure 2.

In the more general case considered here, of two drops with the line joining their centres along the direction of the impressed field, the flow field must be

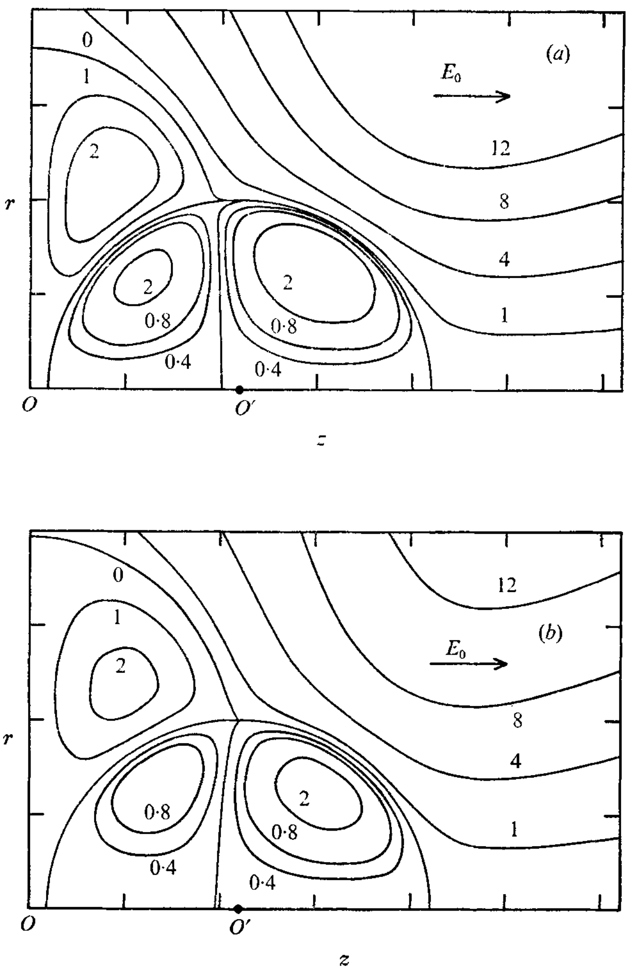

FIGURE 4. Streamlines in and about the drop $\xi_0 = \cosh^{-1} 1{\cdot}1$, in the first quadrant of a meridian section of the r, z plane, for the case $M = 5$. The numbers on the curves are values of $40|\psi|/A$. The centre of the drop is at O'. (a) $\lambda = 10$. (b) $\lambda = 0{\cdot}05$.

computed from (14). We have chosen $M = 5$ and computed ψ for several pairs of λ and ξ_0. Figures 3 and 4 show some of our results. The integrals occurring in (26) were evaluated by means of Simpson's rule. Owing to the oscillatory nature of some of the Legendre polynomials occurring in the integrand we restricted the step length to 0·01. For $\xi_0 \geqslant 1$ the first few terms on the right-hand side of (14) provided a reasonable approximation to ψ, but for programming uniformity we assumed

$$\psi = (\cosh\xi - \mu)^{-\frac{3}{2}} (1-\mu^2)^{\frac{1}{2}} \sum_{n=1}^{28} U_n(\xi) P_n'(\mu) \tag{28}$$

for all data considered. The solution of the problem for one set of data takes about 140 s on the 1907 ICL computer of Sheffield University.

When $\xi_0 = O[\cosh^{-1}(10)]$, that is when the distance between the centres of the drops is of order ten drop diameters, $(p_{\xi\eta})_E$ and the flow field within one drop diameter from the centre of each drop are practically unaffected by the presence of the other drop. As ξ_0 decreases $(p_{\xi\eta})_E$ is modified and so is the flow field. As expected and as shown in figure 1, $(p_{\xi\eta})_E$ is affected more substantially in the region around the points of closest approach of the drop surfaces ($\mu = -1$) than on the surface around $\mu = 1$. The presence of the other drop and the modification of $(p_{\xi\eta})_E$ result in the modification of the flow field in and about the reference drop. Comparison of figures 3 and 4, which show streamlines for the cases $\xi_0 = \cosh^{-1} 1{\cdot}5$ and $\xi_0 = \cosh^{-1} 1{\cdot}1$ for $M = 5$, with figure 2 confirms this. In figures 3 and 4 the structure of the flow field is substantially different from that shown in figure 2. The symmetry of the flow field about the plane half-way between the drops perpendicular to the undisturbed electric field, as expected, is destroyed. The eddies in the region between the drops but outside them become closed and the adjacent eddies remain open and extend up to the plane of symmetry of the system. As the distance between drops having conductivity much higher than the surrounding medium is decreased (see figures 3*a* and 4*a*) the vorticity near the drop surfaces and the open-eddy flow field are enhanced. In the case of drops with low conductivity in comparison with the surrounding fluid (figures 3*b* and 4*b*), $(p_{\xi\eta})_E$ and the intensity of the flow field around the region of the points of closest approach of the drops are substantially reduced if the drops are fairly close together.

The pressure p satisfies Laplace's equation and, apart from an additive constant, for the particular configuration considered here can be expressed as

$$p_1 = (\cosh\xi-\mu)^{\frac{1}{2}} \sum_{n=0}^{\infty} F_n \exp\left[-(n+\tfrac{1}{2})\xi\right] P_n(\mu), \tag{29}$$

$$p_2 = (\cosh\xi-\mu)^{\frac{1}{2}} \sum_{n=0}^{\infty} G_n \cosh(n+\tfrac{1}{2})\xi\, P_n(\mu). \tag{30}$$

The constants F_n and G_n occurring in (29) and (30) can be evaluated from (12) in terms of C_n. From (12) and (13) we obtain

$$(\cosh\xi-\mu)^{\frac{1}{2}}(1-\mu^2)\frac{\partial p}{\partial\mu} = \nu\rho(\cosh\xi-\mu)^{\frac{3}{2}}\frac{\partial}{\partial\xi}[(\cosh\xi-\mu)\nabla^2\psi], \tag{31}$$

where $$\nabla^2 = \frac{\partial}{\partial\xi}(\cosh\xi-\mu)\frac{\partial}{\partial\xi}+(1-\mu^2)\frac{\partial}{\partial\mu}(\cosh\xi-\mu)\frac{\partial}{\partial\mu}.$$

If we now substitute in (31) the appropriate forms of p and ψ [once p_1 and ψ_1 and once p_2 and ψ_2], making use of the relationship

$$(2n+1)\mu P'_n = (n+1)P'_{n-1}+nP'_{n+1}, \tag{32}$$

after some lengthy manipulations we can express both sides of (31) as series in $(1-\mu^2)P'_m$ $(m = 1, 2, 3, \ldots)$. If we then equate coefficients of $(1-\mu^2)P'_m$ on the two sides of the resulting equation we obtain

$$\begin{aligned} F_{m-1}-F_m = {} & (m+1)\{2(m+2)\lambda_{m+1}-[4m-3+(2m+3)e^{2\xi_0}]\lambda_m\} \\ & +(m-1)\{[2m-3+(4m+3)e^{2\xi_0}]\lambda_{m-1}-2(m-2)\lambda_{m-2}e^{2\xi_0}\}, \end{aligned} \tag{33}$$

$$\begin{aligned} G_m-G_{m-1} = {} & 4(m+1)[2(m+2)\mu_{m+1}-(4m-3)\mu_m-(2m+3)s_m]+4(m-1) \\ & \times[(2m-3)\mu_{m-1}+(4m+3)s_{m-1}-2(m-2)\mu_{m-2}] = H_m, \end{aligned} \tag{34}$$

where $\lambda_m = C_m \exp[(m-\frac{1}{2})\xi_0]$, $\mu_m = C_m \sinh[(m+\frac{3}{2})\xi_0]/D_m$

and $s_m = C_m \sinh[(m-\frac{1}{2})\xi_0]/D_m$.

Equations (33) and (34) are valid for $m \geqslant 1$ and $\lambda_{-1} = \mu_{-1} = s_{-1} = 0$. When the velocity field has been calculated the right-hand sides of (33) and (34) are known and thus if we prescribe F_0 and G_0 we can evaluate F_m and G_m for all $m \geqslant 1$. Since for $\xi > 0$,

$$(\cosh\xi - \mu)^{\frac{1}{2}} \Sigma \exp[-(n+\tfrac{1}{2})\xi] P_n(\mu) = 2^{-\frac{1}{2}},$$

it follows from (29) and (33) that F_0 can be arbitrarily specified. In fact, the quantity $2^{\frac{1}{2}}F_0$ represents the pressure at $\xi = \infty$.

Equations (30) and (34) show that we cannot arbitrarily specify G_0. It is obvious from (30) that G_0 must be so chosen that $G_m \to 0$ as $m \to \infty$. We chose $G_0 = 0$ and, from (34), we evaluated G_m $(m \geqslant 1)$ for several sets of data, that is for several sets of ξ_0, λ and M. We found that for every set of data the sequence $G_1, G_2, G_3, \ldots$ quickly converged to some constant, say S, which is dependent on the particular set of data chosen. We must therefore have

$$G_0 = -S = -\sum_{m=1}^{\infty} H_m.$$

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